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Independent sets and non-augmentable paths in arc-locally in-semicomplete digraphs and quasi-arc-transitive digraphs☆

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ABSTRACT

A digraph is arc-locally in-semicomplete if for any pair of adjacent vertices x, y, every in-neighbor of x and every in-neighbor of y either are adjacent or are the same vertex. A digraph is quasi-arc-transitive if for any arc xy, every in-neighbor of x and every out-neighbor of y either are adjacent or are the same vertex. Laborde, Payan and Xuong proposed the following conjecture: Every digraph has an independent set intersecting every non-augmentable path (in particular, every longest path). In this paper, we shall prove that this conjecture is true for arc-locally in-semicomplete digraphs and quasi-arc-transitive digraphs.

1. Introduction and terminology

We only consider finite digraphs without loops and multiple arcs. Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. For any $x, y \in V(D)$, we will write $\overrightarrow{xy}$ or $x \to y$ if $xy \in A(D)$, and also, we will write $\overline{xy}$ if $\overrightarrow{xy}$ or $\overrightarrow{yx}$. For any $u, v, x, y \in V(D)$ if $\overline{uv}$, $\overrightarrow{xu}$ and $\overrightarrow{yv}$, then we will write $\overrightarrow{x}\,\overline{uv}\,\overleftarrow{y}$. For disjoint subsets X and Y of $V(D)$ or subdigraphs of D, $X \to Y$ means that every vertex of X dominates every vertex of Y, $X \Rightarrow Y$ means that there is no arc from Y to X and $X \mapsto Y$ means that both of $X \to Y$ and $X \Rightarrow Y$ hold. For a vertex x in D, its *out-neighborhood* $N^+(x) = \{y \in V(D) : xy \in A(D)\}$ and its *in-neighborhood* $N^-(x) = \{y \in V(D) : yx \in A(D)\}$. For a set $W \subseteq V$, let $N^+(W) = \cup_{w \in W} N^+(w) - W$, $N^-(W) = \cup_{w \in W} N^-(w) - W$. For a pair X, Y of vertex sets of D, define $[X, Y] = \{xy \in A(D) : x \in X, y \in Y\}$. Let D' be a subdigraph of D and $x \in V(D) - V(D')$. We say that x and D' are adjacent if x and some vertex of D' are adjacent. A *strong component* of a digraph D is a maximal induced subdigraph of D which is strong. The *strong component digraph* $SC(D)$ of D is obtained by contracting strong components of D and deleting any parallel arcs obtained in this process. An *empty digraph* is a simple digraph in which no two vertices are adjacent.

A path $P = x_0x_1 \ldots x_k$ in D is *non-augmentable* if there exists no path $y_0y_1 \ldots y_s$ in $D - V(P)$ such that $\overrightarrow{y_sx_0}$ or $\overrightarrow{x_ky_0}$ or $\overrightarrow{x_{i-1}y_0}$ and $\overrightarrow{y_sx_i}$ for some $1 \le i \le k$. Clearly, a longest path must be a non-augmentable path, but the converse is not true. A path $Q = x_0x_1 \ldots x_t$ in D is *internally and initially non-augmentable* if there exists no path $y_0y_1 \ldots y_s$ in $D - V(Q)$ such that $\overrightarrow{y_sx_0}$ or $\overrightarrow{x_{i-1}y_0}$ and $\overrightarrow{y_sx_i}$ for some $1 \le i \le t$. A path P in D intersects a subset F of $V(D)$ if $V(P) \cap F \ne \emptyset$.

A digraph is *arc-locally in-semicomplete (arc-locally out-semicomplete)* if for any pair of adjacent vertices x, y, every in-neighbor (out-neighbor) of x and every in-neighbor (out-neighbor) of y either are adjacent or are the same vertex. A digraph is *quasi-arc-transitive* if for any arc xy, every in-neighbor of x and every out-neighbor of y either are adjacent or are the same

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vertex. A digraph is *quasi-antiarc-transitive* if for any arc xy, every in-neighbor of y and every out-neighbor of x either are adjacent or are the same vertex. For concepts not defined here we refer the reader to [1,2].

In [6], Laborde et al. conjectured that in every digraph, there exists an independent set intersecting every longest path and showed that this conjecture is true for symmetric digraphs. This conjecture is still open. Many classes of digraphs have kernels, such as transitive digraphs. In [4], Galeana-Sánchez and Gómez showed that this conjecture is true for digraphs having a kernel. In [5], Galeana-Sánchez and Rincón-Mejia proved the conjecture for line digraphs, arc-locally semicomplete digraphs, quasi-antiarc-transitive digraphs, quasi-transitive digraphs, path-mergeable digraphs, locally in-semicomplete digraphs, locally out-semicomplete digraphs, semicomplete digraphs and semicomplete k-partite digraphs, all of which are generalizations of tournaments except line digraphs. Note that arc-locally in-semicomplete digraphs, arc-locally out-semicomplete digraphs and quasi-arc-transitive digraphs are also generalizations of tournaments. In this paper, we will prove this conjecture for these three classes of digraphs.

2. Arc-locally in-semicomplete digraphs

Let us start with two classes of digraphs which are closely related to arc-locally in-semicomplete digraphs.

Let C be a cycle of length $k \geq 2$ and let $V_1, V_2, \ldots, V_k$ be pairwise disjoint vertex sets. *The extended cycle* $C[V_1, V_2, \ldots, V_k]$ is the digraph with vertex set $V_1 \cup V_2 \cup \cdots \cup V_k$ and arc set $\cup_{i=1}^{k}\{v_i v_{i+1} : v_i \in V_i, v_{i+1} \in V_{i+1}\}$, where subscripts are taken modulo k. That is, we have $V_1 \mapsto V_2 \mapsto \cdots \mapsto V_k \mapsto V_1$ and there are no other arcs in this extended cycle. Let H_1 and H_3 be two empty digraphs, H_2 be a trivial empty digraph, H_4 be a semicomplete digraph and let H be a digraph with vertex set $V(H_1) \cup V(H_2) \cup V(H_3) \cup V(H_4)$ and arc set $A(H_4) \cup \{uv : u \in V(H_3) \cup V(H_4), v \in V(H_1)\} \cup \{xy : x \in V(H_4), y \in V(H_3)\} \cup \{zw : z \in V(H_2), w \in V(H_3)\}$, where H_1, H_2, H_3 and H_4 are pairwise disjoint and one of $V(H_3)$ and $V(H_4)$ is permitted to be empty. Add some arcs between $V(H_2)$ and $V(H_1) \cup V(H_4)$ to H such that the resulting digraph D is strong and the vertex of H_2 is adjacent to every vertex of $H_1 \cup H_4$. It is easy to see that $H_1 \to H_2$, $H_2 \mapsto H_3$, $H_3 \cup H_4 \mapsto H_1$, $H_4 \mapsto H_3$. Such D is called a *T-digraph* with T-partition $(V(H_1), V(H_2), V(H_3), V(H_4))$.

The following result can be found in [7].

Lemma 2.1 (*[7]*)**.** *Let D be a strong arc-locally in-semicomplete digraph, then D is either a semicomplete digraph, a semicomplete bipartite digraph, an extended cycle or a T-digraph.*

The following lemmas play an important role in our paper.

Lemma 2.2. *Let D be an arc-locally in-semicomplete digraph and let D' be a non-trivial strong subdigraph of D. For any $s \in V(D) - V(D')$, if there exists a path from s to D', then s and D' are adjacent.*

Proof. Let $P = sx_1 \ldots x_k$ be a shortest path from s to D'. We prove that s is adjacent to some vertex in D' by induction on the length k of P. Obviously, the assertion holds when $k = 1$. For any $k \geq 2$, we suppose that the assertion holds for $k - 1$. Note that $x_1 \ldots x_k$ is a path of length $k - 1$. By the induction hypothesis, there exists a vertex $u \in V(D')$ such that u and x_1 are adjacent. Since D' is a non-trivial strong digraph, there exists a vertex $v \in V(D')$ such that $v \to u$. Then $\overline{sv}$ because $\overrightarrow{s}\,\overline{x_1 u}\,\overleftarrow{v}$ and D is arc-locally in-semicomplete. The proof of Lemma 2.2 is complete. □

Lemma 2.3. *Let D' be a subdigraph of an arc-locally in-semicomplete digraph D and let $s \in V(D) - V(D')$ such that there exists an arc from s to D' and $s \Rightarrow D'$. Then each of the following holds:*

(a) *If D' is a path of even length and s dominates the terminal vertex of D', then s dominates the initial vertex of D'.*
(b) *If D' is a cycle and s dominates two consecutive vertices in D', then $s \mapsto D'$.*
(c) *If D' is an odd cycle, then $s \mapsto D'$.*

Proof. (a) Let $D' = x_0 x_1 \ldots x_{2k}$. For $k = 0$ the assertion is trivial, so assume $k \geq 1$. By the hypothesis, $\overrightarrow{x_{2k-2}}\,\overline{x_{2k-1}x_{2k}}\,\overleftarrow{s}$ which implies that $\overline{sx_{2k-2}}$. Combining this with $s \Rightarrow D'$, we have $s \to x_{2k-2}$. Continuing in this way, it follows that $s \to x_0$.

(b) Let $D' = y_1 y_2 \ldots y_t y_1$. Without loss of generality, assume that $s \to \{y_1, y_2\}$. Let $y \in V(D') - \{y_1, y_2\}$ be arbitrary. Note that one of the lengths of $D'[y, y_1]$ and $D'[y, y_2]$ must be even. By (a), $s \to y$. So $s \mapsto D'$ follows from $s \Rightarrow D'$.

(c) Let $D' = z_1 z_2 \ldots z_{2k+1} z_1$. Without loss of generality, assume that $s \to z_{2k+1}$. Note that the length of $D'[z_1, z_{2k+1}]$ is even. By (a), we have that $s \to z_1$. Therefore, it follows from (b) that $s \mapsto D'$. The proof of Lemma 2.3 is complete. □

Lemma 2.4. *Let D be an arc-locally in-semicomplete digraph and let D' be a non-trivial strong induced subdigraph of D and let $s \in V(D) - V(D')$ such that there exists an arc from s to D' and $s \Rightarrow D'$. Then each of the following holds:*

(a) *If D' is a bipartite digraph with bipartition (X, Y) and s dominates a vertex of X, then $s \mapsto X$.*
(b) *If D' is a non-bipartite digraph, then $s \mapsto D'$.*

Proof. (a) Let sx be an arc from s to X. For any $x_1 \in X - \{x\}$, since D' is a strong bipartite digraph and $x, x_1 \in X$, there exists an (x_1, x)-path P of even length. By Lemma 2.3(a), $s \to x_1$. Combining this with $s \Rightarrow D'$, we have $s \mapsto X$.

(b) First we claim that if there exists an odd cycle C in D' such that s dominates some vertex of C, then $s \mapsto D'$. Let $C = x_1x_2 \dots x_{2k+1}x_1$. By Lemma 2.3(c), $s \mapsto C$. For any $x \in V(D')$, let P be a shortest path from x to C and let x_i be the terminal vertex of P. Then one of the lengths of P and Px_i^+ is even, where x_i^+ is the successor of x_i in C. It follows that $s \to x$ from Lemma 2.3(a). By the arbitrariness of x, $s \to D'$ and so $s \mapsto D'$.

Recall that D' is a non-trivial strong induced subdigraph of D and is not a bipartite digraph. It can be that there are at least three vertices in D', and by Lemma 2.1, D' is either a semicomplete digraph, an extension of an odd cycle or a T-digraph. Suppose that D' is either a semicomplete digraph or an extension of an odd cycle. If D' is a semicomplete digraph, then every vertex of D' is on a 3-cycle. If D' is an extension of an odd cycle, then every vertex of D' is on some odd cycle. Let sy be an arc from s to D'. So y is on an odd cycle of D'. By the above claim, we have $s \mapsto D'$. Suppose that D' is a T-digraph with T-partition $(V_1, \{v\}, V_3, V_4)$. Note that every vertex of $V_1 \cup \{v\} \cup V_3$ is on a 3-cycle. If $y \in V_1 \cup \{v\} \cup V_3$, then by the above claim, we have $s \mapsto D'$. Suppose $y \in V_4$. By the definition of T-digraphs, there exists $z \in V_4$ such that $v \to z$. If $z = y$, then $vyxv$, where $x \in V_1$, is a 3-cycle. By the above claim, $s \mapsto D'$. If $z \neq y$, then $\overrightarrow{s}\,\overline{yz}\,\overleftarrow{v}$ implies $\overline{sv}$ and so $s \to v$. Since v is on the 3-cycle $vzxv$, where $x \in V_1$, we have $s \mapsto D'$. The proof of Lemma 2.4 is complete. □

Lemma 2.5. *Let D be an arc-locally in-semicomplete digraph and let D_1 and D_2 be two distinct non-trivial strong components of D with at least one arc from D_1 to D_2. Then either $D_1 \mapsto D_2$ or $D_1 \cup D_2$ is a bipartite digraph. In particular, if $D_1 \cup D_2$ is a bipartite digraph, then D_1 and D_2 are bipartite with bipartitions (X_1, Y_1) and (X_2, Y_2), respectively, and $X_1 \mapsto Y_2$, $Y_1 \mapsto X_2$.*

Proof. Claim A. Every vertex of D_1 is adjacent to D_2.

Let xy be an arc from D_1 to D_2. For any $z \in V(D_1) - \{x\}$, since D_1 is strong, there exists a path P from z to x. Hence Py is a path from z to D_2. Then the assertion follows from Lemma 2.2. The proof of Claim A is complete.

If D_2 is a non-bipartite digraph, then, by Lemma 2.4(b) and Claim A, $D_1 \mapsto D_2$. Now suppose that D_2 is a bipartite digraph. Let (X_2, Y_2) be a bipartition of D_2.

Claim B. Let zx be an arc from D_1 to X_2. For any $z_0 \in V(D_1) - \{z\}$, if there is a (z_0, z)-path of odd length, then $z_0 \mapsto Y_2$; if there is a (z_0, z)-path of even length, then $z_0 \mapsto X_2$.

Let $P = z_0z_1 \dots z_n$ be a (z_0, z)-path, where $z_n = z$. By the strong connectivity of D_2, there exists $y \in Y_2$ such that $y \to x$. So $\overrightarrow{z_{n-1}}\,\overline{z_nx}\,\overleftarrow{y}$, which implies that $\overline{z_{n-1}y}$ and so $z_{n-1} \to y$. By Lemma 2.4(a), $z_{n-1} \mapsto Y_2$. Continuing in this way, it follows that if n is even, then $z_0 \mapsto X_2$; if n is odd, $z_0 \mapsto Y_2$. The proof of Claim B is complete.

It is obvious that D_1 is an arc-locally in-semicomplete digraph. By Lemma 2.1, D_1 is either a semicomplete digraph, a semicomplete bipartite digraph, an extended cycle or a T-digraph. Note that a semicomplete digraph of order 2 is also a semicomplete bipartite digraph. We consider two cases.

Case 1. D_1 is either a semicomplete digraph of order at least 3, a T-digraph or an extension of an odd cycle.

It is not difficult to see that there exists an odd cycle C in D_1. Write $C = x_1x_2 \dots x_{2k+1}x_1$. Consider an arbitrary vertex of C, say x_{2k+1}. By Claim A, x_{2k+1} dominates some vertex of D_2, say y. Without loss of generality, assume $y \in X_2$. By Lemma 2.4(a), $x_{2k+1} \mapsto X_2$. Note that $x_1x_2 \dots x_{2k+1}$ is a path of even length. By Claim B, $x_1 \mapsto X_2$ and again using the path $x_{2k+1}x_1$ and Claim B, we have that $x_{2k+1} \mapsto Y_2$. Hence, $x_{2k+1} \mapsto D_2$ and so $C \mapsto D_2$. For any $x \in V(D_1)$, let P be a shortest path from x to C and let x_i be the terminal vertex of P. By Claim B and $C \mapsto D_2$, we have $x \mapsto D_2$. It follows that $D_1 \mapsto D_2$ from the arbitrariness of x and $C \mapsto D_2$.

Case 2. D_1 is either a semicomplete bipartite digraph or an extension of an even cycle.

In this case D_1 is a bipartite digraph. Let (X_1, Y_1) be a bipartition of D_1. For any $x \in X_1$, by Claim A, x and D_2 are adjacent. Without loss of generality, assume that x and some vertex of Y_2 are adjacent. By Lemma 2.4(a), $x \mapsto Y_2$. By Claim B and the fact that D_1 is a strong bipartite digraph, we have $Y_1 \mapsto X_2$ and $X_1 \mapsto Y_2$. If there exists a vertex of X_1 which dominates some vertex of X_2, then by Claim B, $X_1 \mapsto X_2$ and $Y_1 \mapsto Y_2$. Hence $D_1 \mapsto D_2$. Now we assume that $[X_1, X_2] = \emptyset$. By Claim B, it is not difficult to obtain that $[Y_1, Y_2] = \emptyset$. Thus $D_1 \cup D_2$ is a bipartite digraph with bipartition $(X_1 \cup X_2, Y_1 \cup Y_2)$ and $X_1 \mapsto Y_2$, $Y_1 \mapsto X_2$. The proof of Lemma 2.5 is complete. □

Let H_1 and H_2 be two disjoint subdigraphs of a digraph. Let $l(H_1, H_2) = \min\{\mathrm{dist}(x, y) : x \in V(H_1), y \in V(H_2)\}$.

Lemma 2.6. *Let D be a connected non-strong arc-locally in-semicomplete digraph. If there are more than one initial strong components, then they are trivial.*

Proof. Suppose, on the contrary, that there exists one non-trivial initial strong component, say D_1. Since there are at least two initial strong components, there exists an initial strong component D_2 and a strong component D_3 such that D_3 is reachable from both D_1 and D_2 in D. We choose such a strong component, to simplify notation, denoted by D_3, such that $l(D_1, D_3) + l(D_2, D_3)$ is as small as possible. Let $P = x_0x_1 \dots x_k$ and $Q = z_0z_1 \dots z_s$ be the shortest paths from D_1 to D_3 and from D_2 to D_3, respectively.

First, we show that $k = s = 1$. Suppose that D_3 is non-trivial. It follows from Lemma 2.2 and the minimality of P and Q. Suppose that D_3 is trivial. So $x_k = z_s$. Assume $s \geq 2$. Then $\overline{x_{k-1}z_{s-2}}$ because $\overrightarrow{z_{s-2}}\,\overline{z_{s-1}z_s}\,\overleftarrow{x_{k-1}}$. If $\overrightarrow{x_{k-1}z_{s-2}}$, then z_{s-2} is reachable from D_1 and D_2 and so $l(D_1, z_{s-2}) + l(D_2, z_{s-2}) < l(D_1, D_3) + l(D_2, D_3)$, a contradiction to the choice of D_3. Similarly, if $\overrightarrow{z_{s-2}x_{k-1}}$, then we can also obtain a contradiction. So $s = 1$. In the same way, we can show that $k = 1$. If $x_0 \to z_1$, then since D_1 is a non-trivial strong component, there exists $y \in V(D_1) - \{x_0\}$ such that $y \to x_0$. Then we have $\overrightarrow{y}\,\overline{x_0 z_1}\,\overleftarrow{z_0}$, which implies that $\overline{z_0 y}$, a contradiction to the fact that D_1 and D_2 are initial strong components. Suppose not. By Lemma 2.4, D_3 must be bipartite and x_1 and z_1 belong to different parts. Since D_3 is strong, there is a (z_1, x_1)-path R of odd length. Let w be the successor of z_1 in R. Then the length of $R[w, x_1]$ is even. By Lemma 2.3(a), $x_0 \to w$. So we have $\overrightarrow{z_0}\,\overline{z_1 w}\,\overleftarrow{x_0}$, which implies that $\overline{x_0 z_0}$, a contradiction. The proof of Lemma 2.6 is complete. $\square$

Lemma 2.7. *Let D be a strong arc-locally in-semicomplete digraph. Then there exists an independent set intersecting every internally and initially non-augmentable path in D.*

Proof. By Lemma 2.1, D is either a semicomplete digraph, a semicomplete bipartite digraph, an extension of a cycle or a T-digraph. Clearly, the assertion holds for the case $|V(D)| = 1$. Now assume $|V(D)| \geq 2$. Let $P = x_0 x_1 \ldots x_k$ be an internally and initially non-augmentable path. Then $N^-(x_0) \subseteq V(P)$ and the length of P is at least 1.

Suppose that D is a semicomplete digraph. We claim that P is a Hamiltonian path. Suppose not. Then we have $V(D) - V(P) \neq \emptyset$. Since D is a strong semicomplete digraph, there exists a vertex $x \in V(D) - V(P)$ such that x dominates some vertex of P, say x_i, and x and every vertex of P are adjacent. Combining this with the definition of P, we have that $i \neq 0$ and $\overrightarrow{xx_{i-1}}$. Continuing in this way, it follows that $\overrightarrow{xx_0}$. Hence the claim is true. Note that the independent set in D consists of one vertex. For any $z \in V(D)$, by the above argument, z intersects every internally and initially non-augmentable path in D.

Suppose that D is a semicomplete bipartite digraph with bipartition (X, Y). Since D is a strong semicomplete bipartite digraph, the length of an internally and initially non-augmentable path is at least 1 in D. That is, every internally and initially non-augmentable path contains both some vertices of X and some vertices of Y. Hence X intersects every internally and initially non-augmentable path.

Suppose that D is an extension of a cycle. Let $D = C_k[E_1, E_2, \ldots, E_k]$, where E_i $(1 \leq i \leq k)$ is an independent set. We claim that E_1 intersects every internally and initially non-augmentable path. Suppose not. There exists an internally and initially non-augmentable path $Q = u_0 u_1 \ldots u_t$ such that $V(Q) \cap E_1 = \emptyset$. We will show that $E_2 \cap V(Q) = \emptyset$. Suppose, on the contrary, that there exists $y \in E_2 \cap V(Q)$. By the choice of P and $E_1 \to E_2$, we have $y \neq u_0$. So $y = u_i$ for some $i \geq 1$. Since $N^-(u_i) = E_1$, we have that $u_{i-1} \in E_1$, a contradiction. Hence $E_2 \cap V(Q) = \emptyset$. Continuing in this way, we can obtain that $E_i \cap V(Q) = \emptyset$, for $i = 1, 2, \ldots, k$, a contradiction to $|V(Q)| \geq 2$.

Suppose that D is a T-digraph with T-partition $(V_1, \{v\}, V_3, V_4)$. Recall that V_1 is an independent set. We claim that V_1 intersects every internally and initially non-augmentable path. Suppose, on the contrary, that $R = z_0 z_1 \ldots z_k$ is an internally and initially non-augmentable path such that $V(R) \cap V_1 = \emptyset$. By the definition of T-digraphs, we know that every vertex of V_1 is adjacent to every vertex of $V(D) - V_1$. Hence, for any $x \in V_1$, we have $\overline{xz_0}$ and $z_0 \mapsto x$ by the choice of R. If $x \to z_1$, then $z_0 x z_1 \ldots z_k$ is a path in D contradicting the fact that R is an internally and initially non-augmentable path. So $z_1 \mapsto x$. Continuing in this way, we can deduce that $z_i \mapsto x$ for every $0 \leq i \leq k$. By the definition of T-digraphs, we have $v \notin V(R)$. Since D is strong, there exists a path from v to R. Let $v v_1 \ldots v_l$ be the shortest path from v to R and $v_l = z_j$. Clearly, $z_j \neq z_0$. Note that $z_0 \ldots z_{j-1} x v v_1 \ldots v_l z_{j+1} \ldots z_k$ is a path, a contradiction. Hence the claim is true and the proof of Lemma 2.7 is complete. $\square$

Since a non-augmentable path must be an internally and initially non-augmentable path, we have the following theorem from Lemma 2.7.

Theorem 2.8. *Let D be a strong arc-locally in-semicomplete digraph. Then there exists an independent set intersecting every non-augmentable path in D.*

Theorem 2.9. *Let D be a connected arc-locally in-semicomplete digraph. Then there exists an independent set intersecting every non-augmentable path in D.*

Proof. If D is strong, then the assertion follows from Theorem 2.8. Next suppose that D is not strong and let $D_0, D_1, \ldots, D_t$ be its strong components. Let $P = x_0 x_1 \ldots x_k$ be a non-augmentable path in D and let $x_0 \in V(D_j)$ with $0 \leq j \leq t$.

Claim A. If D_j is not an initial strong component, then D_j is a bipartite digraph.

Since D_j is not an initial strong component, $N^-(V(D_j)) \neq \emptyset$. For any $x \in N^-(V(D_j))$, by the definition of strong components and $x_0 \in V(D_j)$, we have $x \notin V(P)$, otherwise x and x_0 are in the same component, a contradiction. If $|V(D_j)| = 1$, then $x \to x_0$ and so xP is a longer path than P, a contradiction. Hence, $|V(D_j)| \geq 2$, that is, D_j is a non-trivial strong component. If D_j is a non-bipartite digraph, then, by Lemma 2.4(b), we have $x \mapsto D_j$ and so $x \to x_0$, a contradiction. Hence D_j is a bipartite digraph. The proof of Claim A is complete.

We consider two cases.

Case 1. There exists a non-trivial initial strong component.

By Lemma 2.6, there is only one initial strong component in D, say D_0. First we claim that $j = 0$. Suppose, on the contrary, that $j \geq 1$. By Claim A, D_j is a bipartite digraph. Since D_0 is the unique initial strong component, D_j is reachable from D_0. By Lemma 2.2, there is an arc from D_0 to D_j. By $|V(D_0)| \geq 2$, $|V(D_j)| \geq 2$ and Lemma 2.5, we have that either $D_0 \mapsto D_j$ or $D_0 \cup D_j$ is a bipartite digraph. Suppose $D_0 \mapsto D_j$. Combining this with $V(P) \cap N^-(V(D_j)) = \emptyset$, we can find a longer path than P, a contradiction. So assume that $D_0 \cup D_j$ is bipartite. Since D_0 is non-trivial, we know that D_0 is also a bipartite digraph. Let (X_0, Y_0) and (X_j, Y_j) be the bipartition of D_0 and D_j, respectively. According to Lemma 2.5, we have $X_0 \mapsto Y_j$ and $Y_0 \mapsto X_j$.

Without loss of generality, assume that $x_0 \in X_j$. By $Y_0 \mapsto X_j$, we can also obtain a contradiction. Hence $x_0 \in V(D_0)$. Note that the intersection of D_0 and a non-augmentable path of D is an internally and initially non-augmentable path in D_0. By Lemma 2.7, there exists an independent set F intersecting every internally and initially non-augmentable path in D_0 and so F intersects every non-augmentable path of D.

Case 2. The initial strong components are all trivial.

Let $L^+(D) = \{x \in V(D)$: There exists a non-augmentable path in D starting at $x\}$. First we claim that $L^+(D)$ is an independent set. If $|L^+(D)| = 1$, then the claim is trivial. Next assume that $|L^+(D)| \geq 2$. Suppose, on the contrary, that there exists a pair of vertices x, y in $L^+(D)$ such that $\overline{xy}$. Without loss of generality, assume $\overrightarrow{xy}$. Let P be a non-augmentable path starting at y. By $\overrightarrow{xy}$ and the definition of P, we have that $x \in V(P)$. Note that x is reachable from y and y is reachable from x. Therefore, x and y belong to the same strong component, say D_j. Since the initial strong components are trivial and $x, y \in V(D_j)$, we have that D_j is not an initial strong component. By Claim A, D_j is a bipartite digraph. Since $x \to y$, we know that x and y must belong to different parts of D_j. Using $N^-(V(D_j)) \neq \emptyset$ and Lemma 2.4(a), there exists a vertex $u \in N^-(V(D_j))$ such that $u \to x$ or $u \to y$, a contradiction. Hence, $L^+(D)$ is an independent set. Combining this with the definition of $L^+(D)$, we have that the assertion is true. The proof of Theorem 2.9 is complete. □

Since the converse of an arc-locally in-semicomplete digraph is an arc-locally out-semicomplete digraph and the converse of a non-augmentable path is still a non-augmentable path, we have the following theorem.

Theorem 2.10. *Let D be an arc-locally out-semicomplete digraph. Then there exists an independent set intersecting every non-augmentable path in D.*

3. Quasi-arc-transitive digraphs

We begin this section with a useful lemma.

Lemma 3.1 ([3]). *Let D be a quasi-arc-transitive digraph. For a pair x, y of $V(D)$, if there exists an (x, y)-path of odd length, then x and y are adjacent.*

Let H_n be the digraph with vertex set $\{x_1, x_2, \ldots, x_n\}$ and arc set $\{x_1x_2, x_2x_3, x_3x_1\} \cup \{x_1x_i, x_ix_2 : i = 4, \ldots, n\}$, where $n \geq 4$.

Theorem 3.2 ([3]). *Let D be a strong quasi-arc-transitive digraph of order n. Then D is either a semicomplete digraph, a semicomplete bipartite digraph or isomorphic to H_n.*

Lemma 3.3. *Let D' be a non-trivial strong induced subdigraph of a quasi-arc-transitive digraph D and let $s \in V(D) - V(D')$ with at least one arc from s to D' and $s \Rightarrow D'$. Then each of the following holds:*

(a) *If D' is a bipartite digraph with bipartition (X, Y) and s dominates a vertex of X, then $s \mapsto X$.*
(b) *If D' is a non-bipartite digraph, then $s \mapsto D'$.*

Proof. (a) By Theorem 3.2, D' is a semicomplete bipartite digraph. Let sx be an arc from s to X. For any $x' \in X$, since D' is a strong semicomplete bipartite digraph, there is an (x, x')-path P of even length. Then sP is an (s, x')-path of odd length. By Lemma 3.1, s and x' are adjacent. Combining this with $s \Rightarrow D'$, we have $s \mapsto x'$. By the arbitrariness of x', we have $s \mapsto X$.

(b) By the hypothesis and Theorem 3.2, D' is a semicomplete digraph of order at least 3 or isomorphic to H_n with $n \geq 4$ where n is the order of D'. Suppose that D' is semicomplete. Let sy be an arc from s to D'. Since D' is a strong semicomplete digraph of order at least 3, we can deduce that y must be on a 3-cycle, say $C = y_1y_2y_3y_1$, where $y = y_1$. Then $s \to y_1 \to y_2 \to y_3$ implies that $\overline{sy_3}$ and $s \mapsto y_3$ by $s \Rightarrow D'$. Again by $s \to y_3 \to y_1 \to y_2$, we have $\overline{sy_2}$ and so $s \mapsto y_2$. Hence, $s \mapsto C$. For any $z \in V(D')$, since D' is strong, there exists a path from C to z. Let P be the shortest path from C to z and without loss of generality, assume that y_1 is the initial vertex of P. Note that one of the lengths of sP and sy_3P is odd. By Lemma 3.1, we have $\overline{sz}$ and so $s \mapsto z$. By the arbitrariness of z and $s \Rightarrow D'$, we have that $s \mapsto D'$. Next suppose that D' is isomorphic to H_n. Let the vertex set of D' be $\{y_1, y_2, \ldots, y_n\}$ and the arc set be $\{y_1y_2, y_2y_3, y_3y_1\} \cup \{y_1y_i, y_iy_2 : i = 4, \ldots, n\}$, where $n \geq 4$.

First we claim that if s dominates one of $\{y_1, y_2, y_3\}$, then $s \mapsto D'$. By a similar argument to the above, we can obtain that $s \mapsto \{y_1, y_2, y_3\}$. For any $i \geq 4$, $s \to y_3 \to y_1 \to y_i$ implies $\overline{sy_i}$ and so $s \mapsto y_i$. Hence $s \mapsto D'$.

Let sy be an arc from s to D'. If $y \in \{y_1, y_2, y_3\}$, then, by the above claim, $s \mapsto D'$. If $y = y_i$ for some $i \geq 4$, then $s \to y_i \to y_2 \to y_3$ implies $\overline{sy_3}$ and so $\overrightarrow{sy_3}$. By the above claim, $s \mapsto D'$. The proof of Lemma 3.3 is complete. □

Since the converse of a quasi-arc-transitive digraph is still a quasi-arc-transitive digraph, we have the following lemma.

Lemma 3.4. *Let D' be a non-trivial strong induced subdigraph of a quasi-arc-transitive digraph D and let $s \in V(D) - V(D')$ with at least one arc from D' to s and $D' \Rightarrow s$. Then each of the following holds:*

(a) *If D' is a bipartite digraph with bipartition (X, Y) and there exists a vertex of X which dominates s, then $X \mapsto s$.*
(b) *If D' is a non-bipartite digraph, then $D' \mapsto s$.*

Lemma 3.5. *Let D' be a non-trivial strong subdigraph of a quasi-arc-transitive digraph D. For any $s \in V(D) - V(D')$, if there exists a directed path between s and D', then s and D' are adjacent.*

Proof. Since the converse of a quasi-arc-transitive digraph is still a quasi-arc-transitive digraph, without loss of generality, assume that there exists a path from s to D'. Let $P = sx_1 \ldots x_k$ be a shortest path from s to D'. If k is odd, then, by Lemma 3.1, we have $\overline{sx_k}$. Suppose that k is even. Since D' is a non-trivial strong digraph, there exists $u \in V(D') - \{x_k\}$ such that $x_k \to u$. Since Pu is an (s, u)-path of odd length, we have $\overline{su}$ from Lemma 3.1. The proof of Lemma 3.5 is complete. □

Lemma 3.6. *Let D_1 and D_2 be two distinct non-trivial strong components of a quasi-arc-transitive digraph with at least one arc from D_1 to D_2. Then either $D_1 \mapsto D_2$ or $D_1 \cup D_2$ is a semicomplete bipartite digraph.*

Proof. Claim A. Every vertex of D_1 is adjacent to D_2.

Let xy be an arc from D_1 to D_2. For any $z \in V(D_1)$, since D_1 is strong, there exists a (z, x)-path P. Hence Py is a path from z to D_2. By Lemma 3.5, z is adjacent to D_2. By the arbitrariness of z, the claim is true.

If at least one of D_1 and D_2 is non-bipartite, then, by Lemmas 3.3(b), 3.4(b) and 3.5, we have $D_1 \mapsto D_2$. Next suppose that D_1 and D_2 are both bipartite. By Theorem 3.2, D_1 and D_2 are both semicomplete bipartite digraphs with bipartition (X_1, Y_1) and (X_2, Y_2), respectively.

Claim B. If there exists $x \in V(D_1)$ such that $x \mapsto D_2$, then $D_1 \mapsto D_2$.

Without loss of generality, assume that $x \in X_1$. For any $x' \in X_1$, since D_1 is a strong bipartite digraph, there is an (x', x)-path P_1 of even length. Combining this with $x \mapsto D_2$, there is a path of odd length from x' to every vertex of D_2. By Lemma 3.1, x' and every vertex of D_2 are adjacent and so $x' \mapsto D_2$. By the arbitrariness of x', $X_1 \mapsto D_2$. For any $y \in Y_1$, since D_1 is a strong bipartite digraph, there is a (y, x)-path P_2 of odd length. Let $u \in V(D_2)$ be arbitrary. Since D_2 is a strong bipartite digraph, there exists $v \in V(D_2)$ such that $v \to u$. Since $x \mapsto D_2$, we have $x \to v$. Note that P_2vu is a (y, u)-path of odd length. Combining this with Lemma 3.1, we have that $\overline{yu}$ and so $y \mapsto u$. By the arbitrariness of u, we have $y \mapsto D_2$. Again by the arbitrariness of y, we have $Y_1 \mapsto D_2$. Hence $D_1 \mapsto D_2$. The proof of Claim B is complete.

Let xu be an arc from D_1 to D_2. Without loss of generality, assume that $x \in X_1$ and $u \in Y_2$. By Lemmas 3.3(a) and 3.4(a), we have $X_1 \mapsto Y_2$. If $[X_1, X_2] \neq \emptyset$, then, again by Lemmas 3.3(a) and 3.4(a), we have $X_1 \mapsto X_2$. So $x \mapsto D_2$. By Claim B, we have $D_1 \mapsto D_2$. Next, assume $[X_1, X_2] = \emptyset$. For any $y \in Y_1$, by Claim A, y and D_2 are adjacent. If y and Y_2 are adjacent, then, by Lemma 3.3(a), $y \to u$. For any $v \in X_2$, since D_2 and D_1 are strong, there exists a (u, v)-path P_2 of odd length in D_2 and an (x, y)-path P_1 of odd length in D_1. Note that P_1P_2 is an (x, v)-path of odd length. Combining this with Lemma 3.1 we have $\overline{xv}$, a contradiction. Hence, y and Y_2 are not adjacent and y and X_2 are adjacent. By the arbitrariness of y and Lemma 3.3(a), we have that $Y_1 \mapsto X_2$ and $[Y_1, Y_2] = \emptyset$. Combining these with the fact that D_1 and D_2 are both semicomplete bipartite, we have that $D_1 \cup D_2$ is semicomplete bipartite. The proof of Lemma 3.6 is complete. □

Proposition 3.7 ([5]). *Let D be a semicomplete digraph. Then every maximal independent set consists of a single vertex and intersects every non-augmentable path in D.*

Proposition 3.8 ([5]). *Let D be a semicomplete k-partite digraph. Then every maximal independent set intersects every non-augmentable path in D.*

By the definition of the digraph H_n, it is easy to see that $L = \{x_3, x_4, \ldots, x_n\}$ is the unique maximal independent set in H_n. Let R be a non-augmentable path in H_n and let x be the initial vertex of R. If $x = x_1$, then, since $x_3 \to x_1, x_3 \in V(R)$. If $x = x_3$, then, since $x_4 \to x_2, x_4 \in V(R)$. Hence, L intersects R. Combining this with Theorem 3.2, Propositions 3.7 and 3.8, we have the following theorem.

Theorem 3.9. *Let D be a strong quasi-arc-transitive digraph. Then every maximal independent set intersects every non-augmentable path in D.*

Theorem 3.10. *Let D be a quasi-arc-transitive digraph. Then there exists an independent set intersecting every non-augmentable path in D.*

Proof. If D is strong, then the assertion follows from Theorem 3.9. Therefore, assume that D is not strong and let $D_0, D_1, \ldots, D_k$ be its strong components. Let $D_0, D_1, \ldots, D_s$ be the initial strong components and let F_i be the maximal independent set of D_i, for $i = 0, 1, \ldots, s$. Let $W = \{x \in V(D)$: There exists a non-augmentable path in D starting at x and $x \notin \cup_{i=0}^{s} V(D_s)\}$.

Claim A. W is an independent set or an empty set.

If $|W| \leq 1$, then the claim is obviously true. Now assume $|W| \geq 2$. Suppose, on the contrary, that there exists a pair x, y of W such that $\overline{xy}$. By the definitions of strong components and W, x and y must belong to the same strong component, say D_j. By the definition of W, we have $|N^-(V(D_j))| \geq 1$. Let $u \in N^-(V(D_j))$. If D_j is non-bipartite, then, by Lemma 3.3(b), $u \mapsto D_j$ and so $u \mapsto x$, a contradiction. If D_j is bipartite, then x and y must belong to different parts. Hence, by Lemma 3.3(a), u and one of x and y are adjacent, a contradiction. The proof of Claim A is complete.

Let $F = F_0 \cup \cdots \cup F_s \cup W$. It is not difficult to deduce that F is an independent set in D from Claim A and the definition of F_i, $i = 0, \ldots, s$. Let $P = x_0x_1 \ldots x_k$ be a non-augmentable path in D. Suppose that x_0 is in an initial strong component, say D_0.

Let x_i be the last vertex on P that belongs to D_0. Note that $x_0 \dots x_i$ is an internally and initially non-augmentable path of D_0. By Theorem 3.2, D_0 is either semicomplete, semicomplete bipartite or isomorphic to H_n. If D_0 is semicomplete, then, similar to the second paragraph of the proof of Lemma 2.7, we can deduce that $x_0 \dots x_i$ is a Hamiltonian path of D_0. So F_0 intersects P and F intersects P. If D_0 is semicomplete bipartite, then F_0 is some part of D_0. So F_0 interests $x_0 \dots x_i$ and F intersects P. If D_0 is isomorphic to H_n, then, by the definition of H_n, F_0 intersects P and so F intersects P. Suppose that x_0 is not in any initial strong component. By the definition of W, we have that W intersects P. The proof of Theorem 3.10 is complete. □

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